03-14-2022

Dynamic Sampling Strategy for Morris' Method of Elementary Effects

Franziska Henze, franziska.henze@kit.edu

Institute of Measurement and Control Systems (MRT)
Karlsruhe Institute of Technology (KIT)

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Audi intelligence
automated driving

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Would you take a ride?

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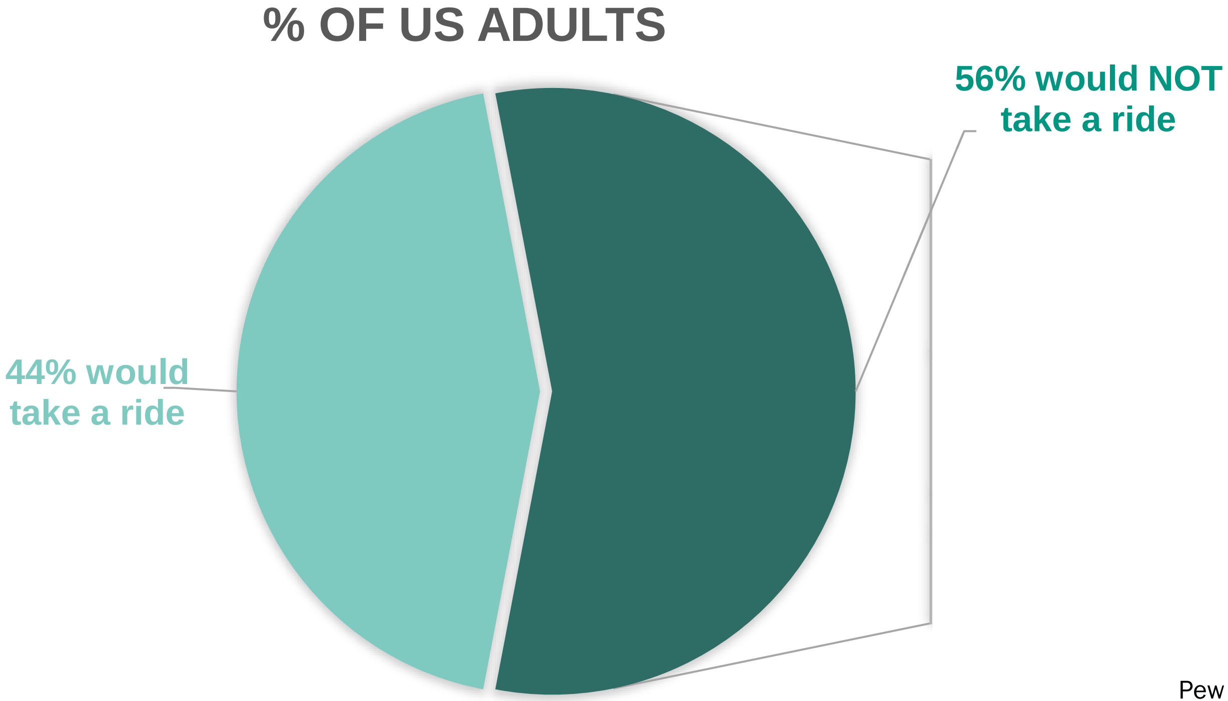
**Franziska Henze (KIT), Markus Rußer (HS Kempten), Dennis Faßbender (Audi),
Christoph Stiller (KIT), Stefan-Alexander Schneider (HS Kempten)**

10th International Conference on Sensitivity Analysis of Model Output | Tallahassee, FL, United States

Would you want to ride in a driverless vehicle?



Pew Research Center, October, 2017, “Automation in Everyday Life”

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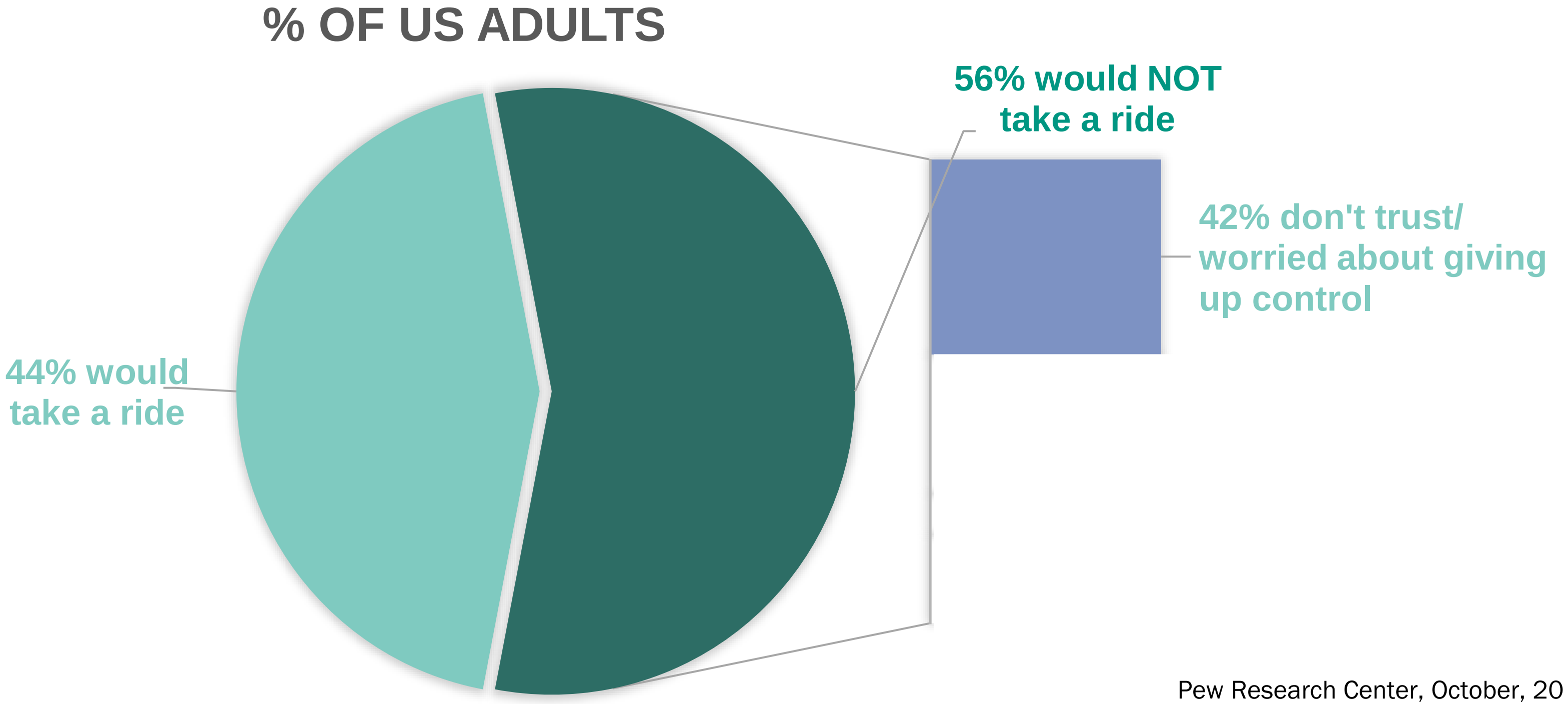

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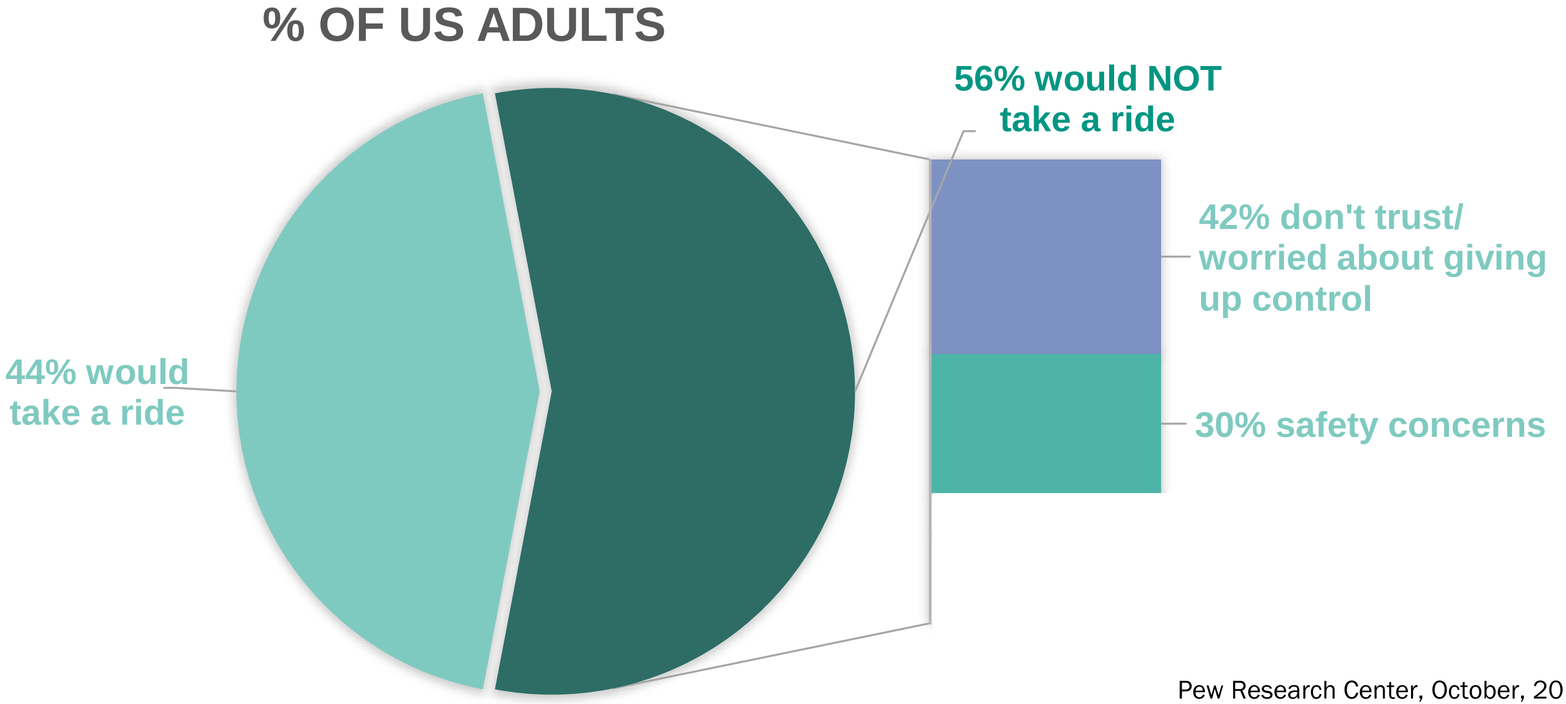

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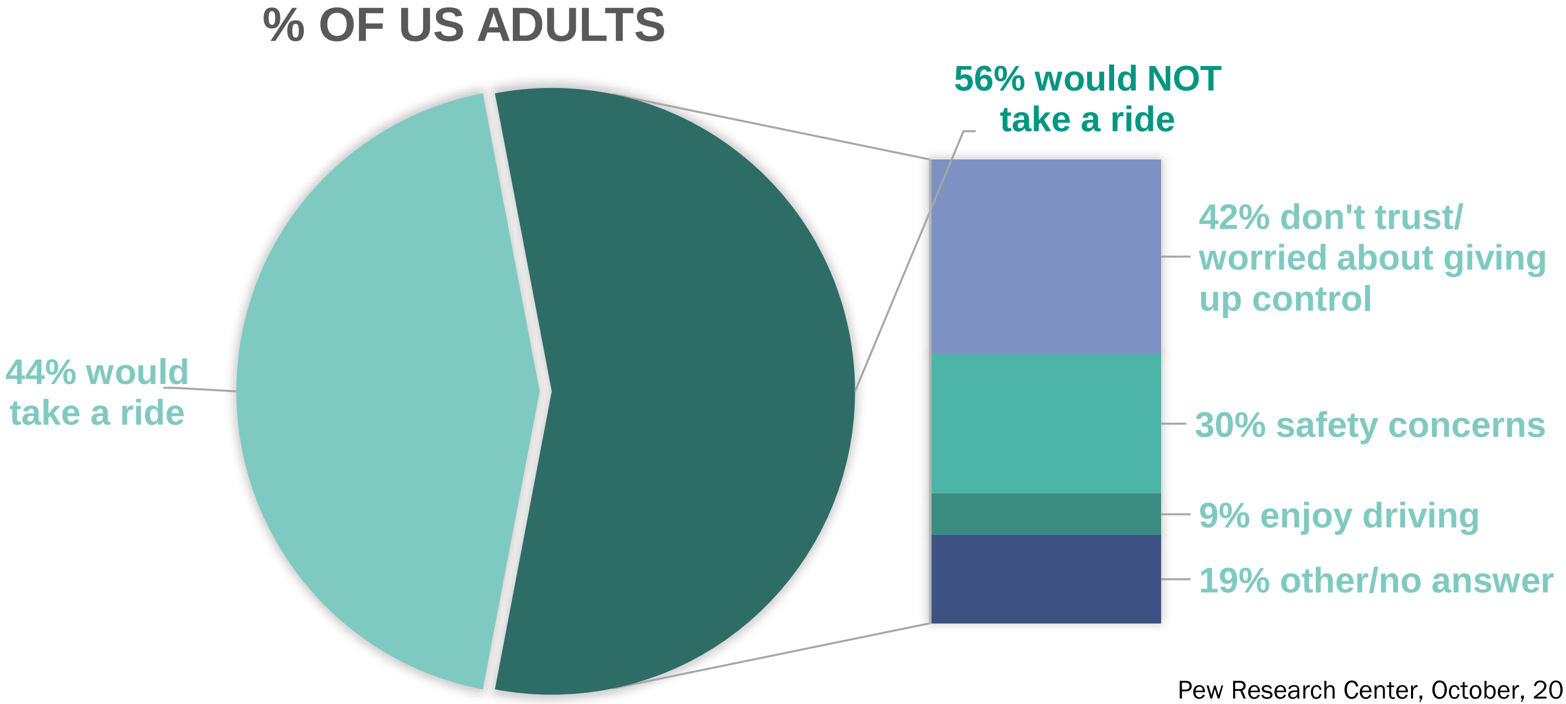

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include relevant information only

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include relevant information only

→ Which information is *relevant*?

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include relevant information only

- → Which information is *relevant*?
- → Morris‘ method of elementary effects!

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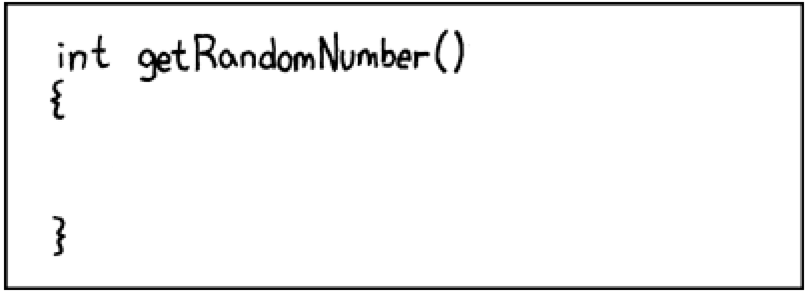
Content



```
int getRandomNumber()
{


}
```

adapted from https://xkcd.com/221/

Morris‘ method: elementary effects (Morris, 1991)

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$$\begin{aligned} \boldsymbol{f}\colon \Omega \subseteq \mathbb{R}^k &\to \mathbb{R}^m \\ \boldsymbol{x} &\mapsto \boldsymbol{f}(\boldsymbol{x}) \end{aligned}$$

Morris‘ method: elementary effects (Morris, 1991)

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elementary effects (for all inputs $i \in \{1, \ldots, k\}$, $k \in \mathbb{N}$), offset $\Delta_i \in \mathbb{R}$

$$\boldsymbol{d}_i(\boldsymbol{x}) \coloneqq \frac{\boldsymbol{f}(\boldsymbol{x} + \Delta_i \boldsymbol{e}_i) - \boldsymbol{f}(\boldsymbol{x})}{\Delta_i}$$

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✓ deterministic $\boldsymbol{f}$, $\boldsymbol{x}$

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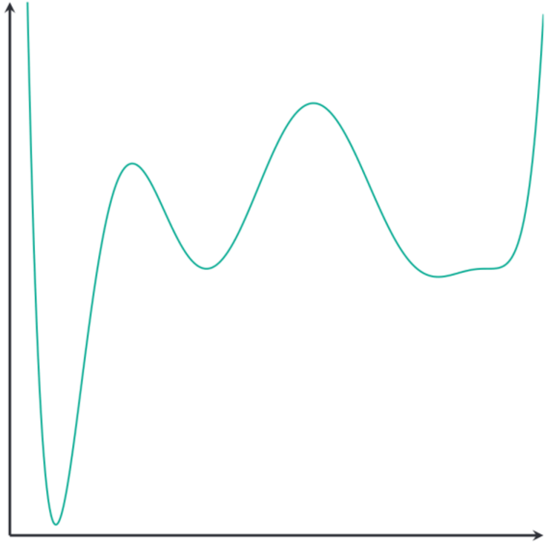
✓ deterministic $\boldsymbol{f}$, $\boldsymbol{x}$

✗ stochastic $\boldsymbol{x}$?

What do elementary effects look like?

- assume $k = 1$, $x \in \mathbb{R}$ to be stochastic
- calculate elementary effects $d_1(x_j)$ for $M = 4$ samples $\{x_j\}_{j=1}^{4}$



reminder:

$$\boldsymbol{d}_i(\boldsymbol{x}_j) = \frac{\boldsymbol{f}(\boldsymbol{x}_j + \Delta_i \boldsymbol{e}_i) - \boldsymbol{f}(\boldsymbol{x}_j)}{\Delta_i}$$

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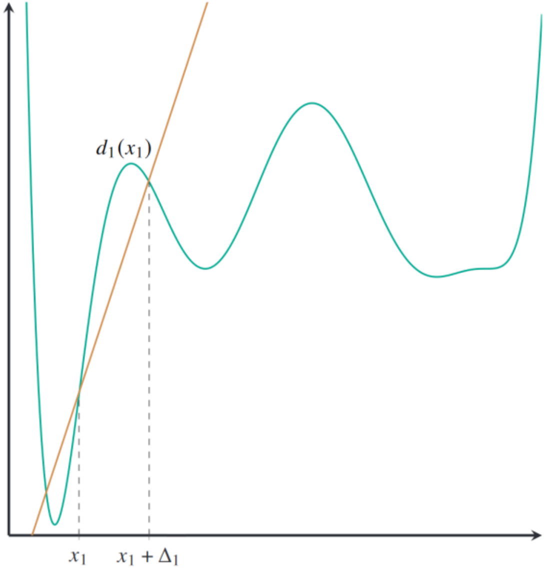

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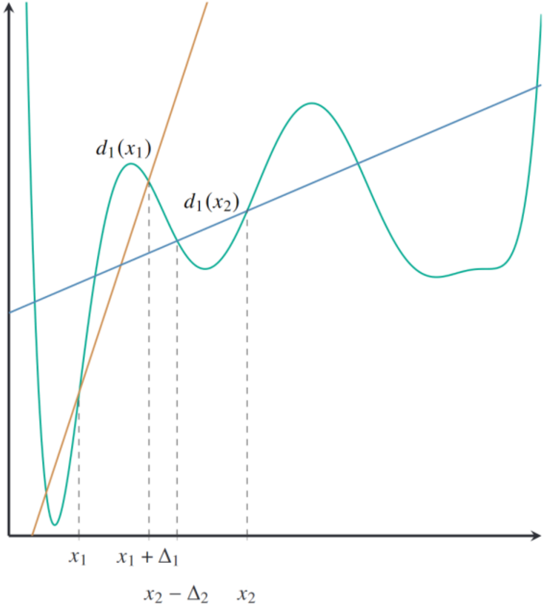

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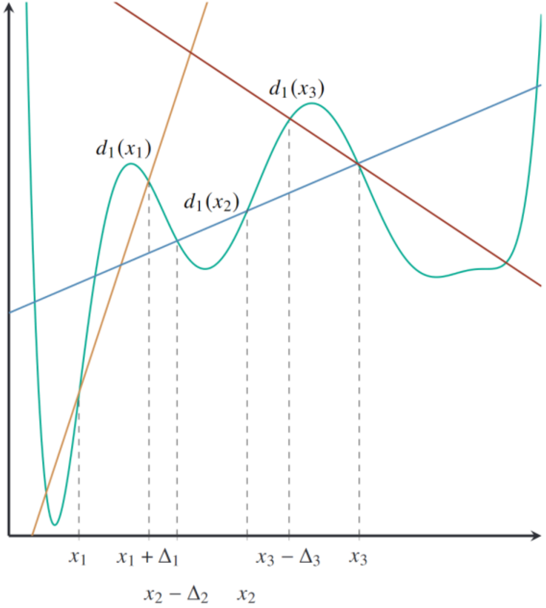

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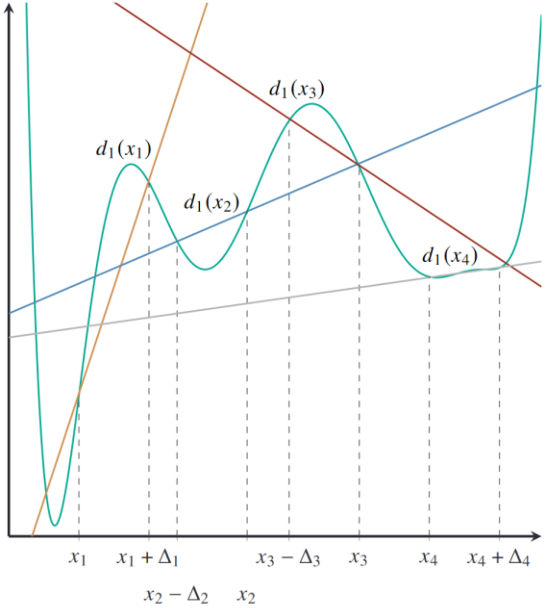

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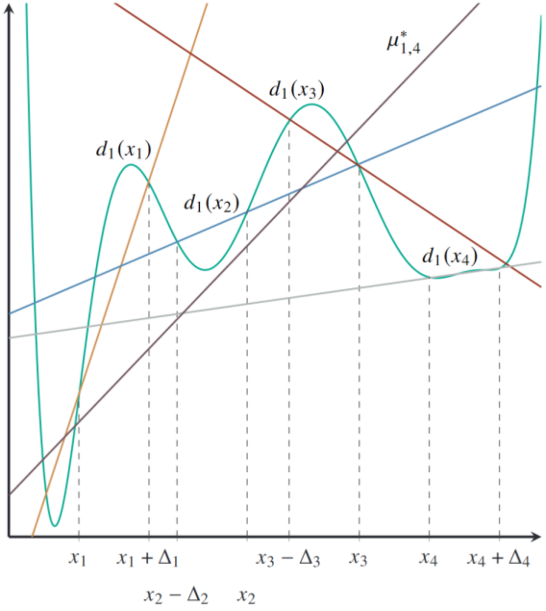

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How can we identify relevant parameters?

- for a set $\{\boldsymbol{x}_j\}_{j=1}^{M}$ of $M \in \mathbb{N}$ samples and for all inputs $i \in \{1, \dots, k\}$
  (Campolongo *et al.*, 2007)

$$\boldsymbol{\mu}_{i,M}^{*} := \sum_{j=1}^{M} \frac{\left|\boldsymbol{d}_i(\boldsymbol{x}_j)\right|}{M}$$

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yes

input i relevant

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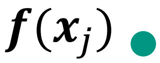
$\left\|\boldsymbol{\mu}_{i,M}^{*}\right\|$ large?

no → input i irrelevant

yes → input i relevant

Number of samples

- elementary effects for $k = 3$:



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Number of samples

- elementary effects for $k = 3$:

$$\boldsymbol{f}(\boldsymbol{x}_j) \bullet\!\!\longrightarrow \boldsymbol{f}(\boldsymbol{x}_j + \Delta_1 \boldsymbol{e}_1)$$

reminder:

$$\boldsymbol{d}_i(\boldsymbol{x}_j) = \frac{\boldsymbol{f}(\boldsymbol{x}_j + \Delta_i \boldsymbol{e}_i) - \boldsymbol{f}(\boldsymbol{x}_j)}{\Delta_i}$$

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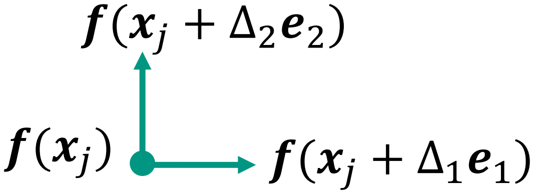

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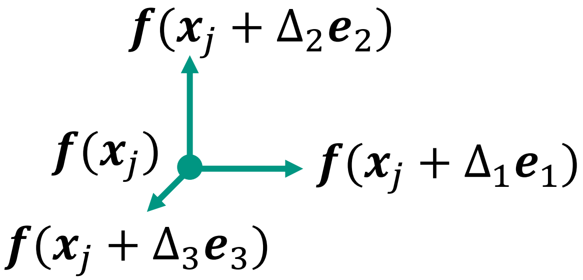

$$f(x_j + \Delta_2 e_2)$$

$$f(x_j)$$

$$f(x_j + \Delta_1 e_1)$$

$$f(x_j + \Delta_3 e_3)$$

reminder:

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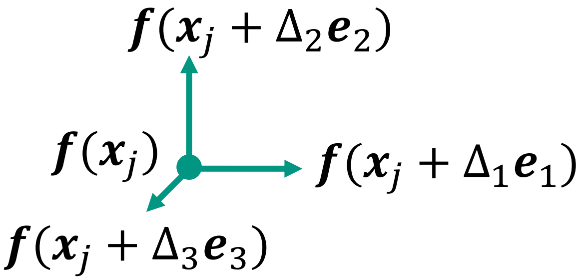

$\boldsymbol{f}(\boldsymbol{x}_j + \Delta_2 \boldsymbol{e}_2)$

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$\boldsymbol{f}(\boldsymbol{x}_j + \Delta_3 \boldsymbol{e}_3)$

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- $4 = k + 1$ function evaluations per elementary effect

Number of samples

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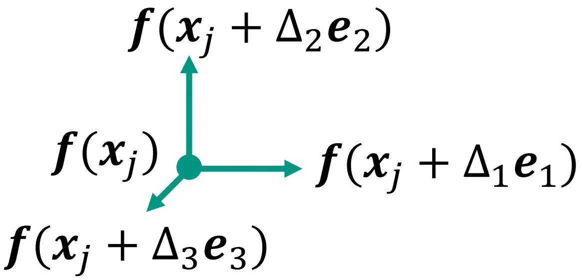

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- $4 = k + 1$ function evaluations per elementary effect
- for M samples: $M(k + 1)$ function evaluations

Number of samples

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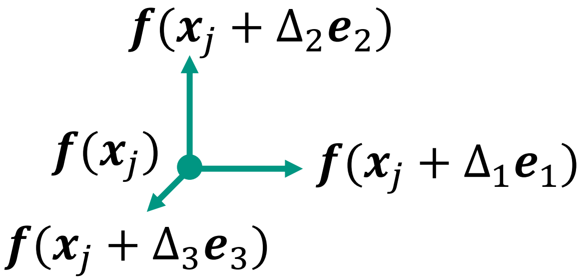

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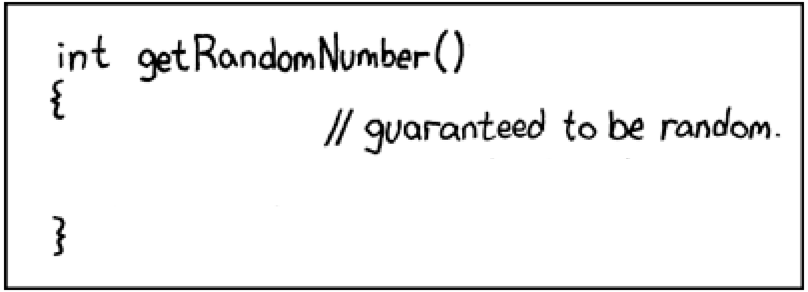
- for M samples: $M(k+1)$ function evaluations

  → curse of dimensionality ☹

Content



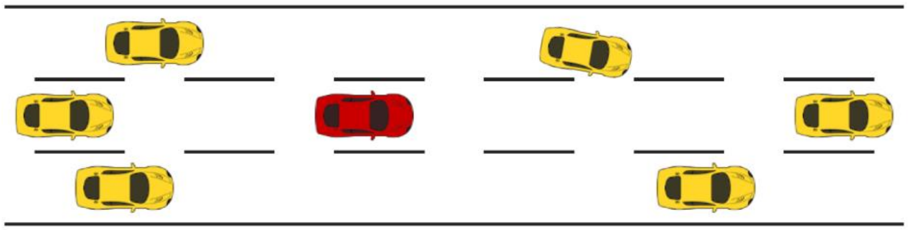
```
int getRandomNumber()
{
                    // guaranteed to be random.


}
```

adapted from https://xkcd.com/221/

How to alleviate the curse of dimensionality?

- example: automated driving



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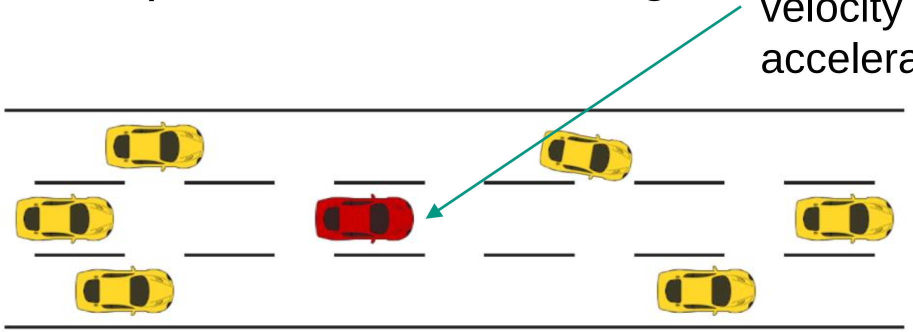

position $\boldsymbol{p}$
velocity $\boldsymbol{v}$
acceleration $\boldsymbol{a}$

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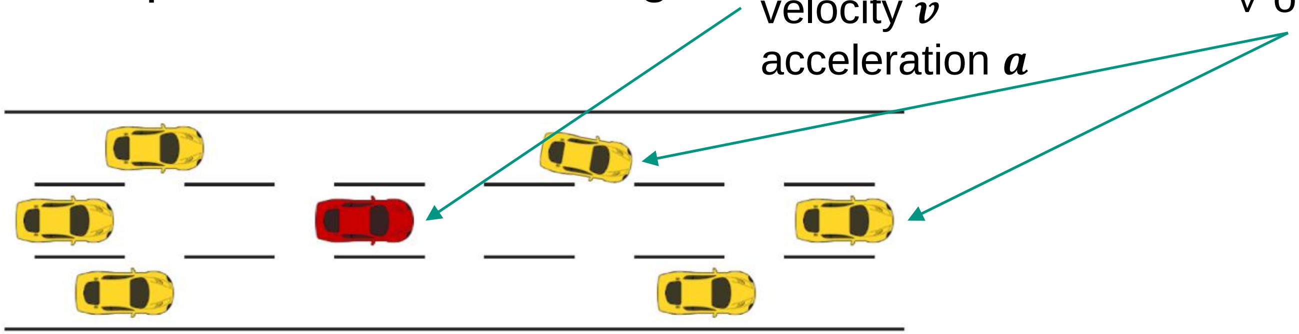

position $\boldsymbol{p}$
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∀ other vehicles:
position $\boldsymbol{p}$
velocity $\boldsymbol{v}$
heading angle φ

How to alleviate the curse of dimensionality?

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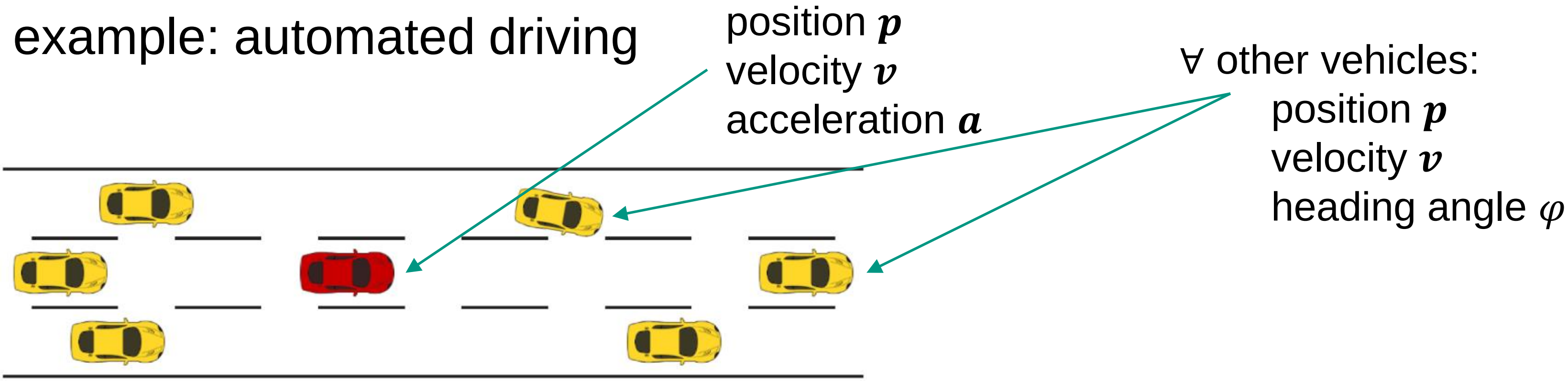

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- many inputs

How to alleviate the curse of dimensionality?

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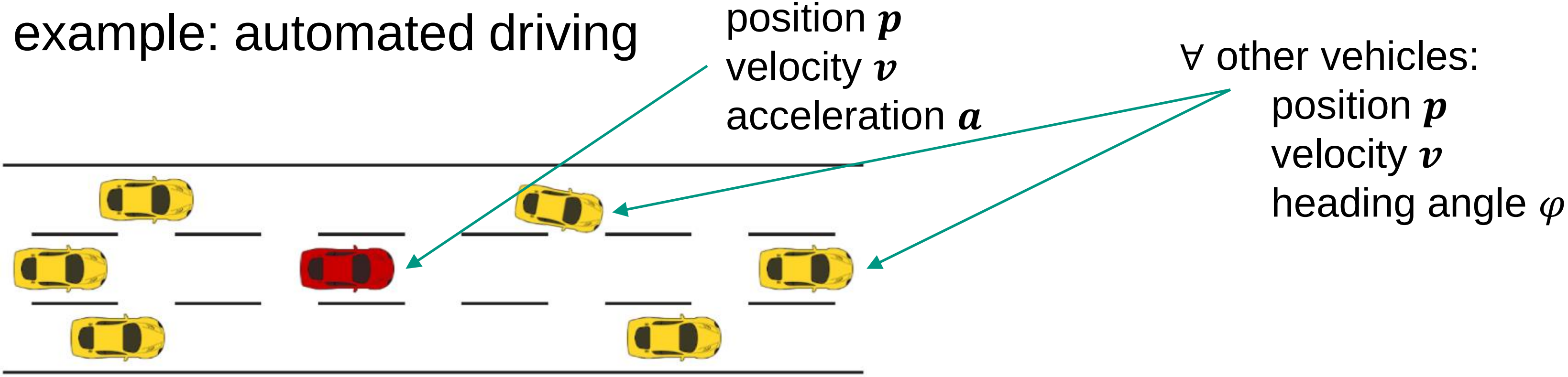

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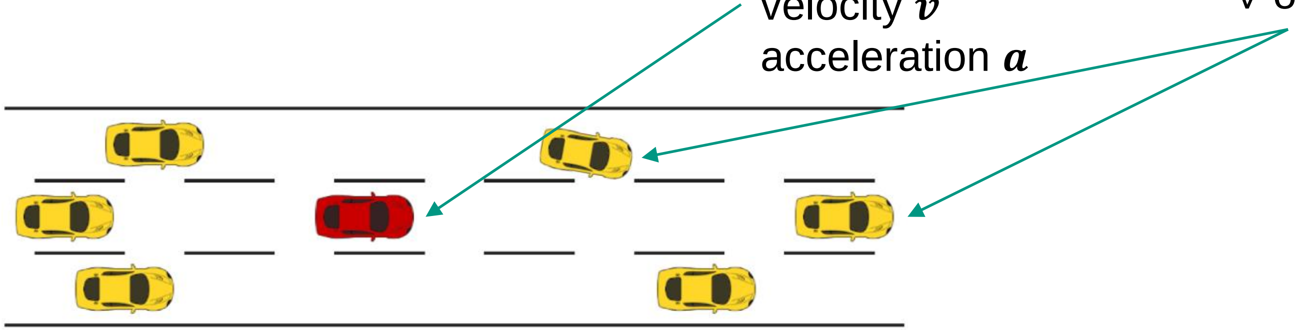
  → curse of dimensionality ⇒ long runtime for M samples

How to alleviate the curse of dimensionality?

- example: automated driving



position $\boldsymbol{p}$
velocity $\boldsymbol{v}$
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∀ other vehicles:
position $\boldsymbol{p}$
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heading angle φ

- many inputs
  - → curse of dimensionality ⇒ long runtime for M samples
  - → instead: choose number of samples M_i for each input i separately

Dynamic Stop Criterion

$$\epsilon^2_{i,M_i} := \frac{1}{10}\sum_{l=1}^{10}\frac{\left\|\boldsymbol{\mu}^*_{i,M_{i-l}} - \boldsymbol{\mu}^*_{i,M_i}\right\|^2}{\left\|\boldsymbol{\mu}^*_{i,M_i}\right\|^2}$$

reminder:

$$\boldsymbol{d}_i(\boldsymbol{x}_j) = \frac{\boldsymbol{f}(\boldsymbol{x}_j + \Delta_i \boldsymbol{e}_i) - \boldsymbol{f}(\boldsymbol{x}_j)}{\Delta_i}$$

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Dynamic Stop Criterion

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$$\epsilon_{i,M_i}^2 < \kappa_{act}$$

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$$\epsilon_{i,M_i}^2 < \kappa_{act}$$

yes

stop evaluating input i
(inactive)

reminder:

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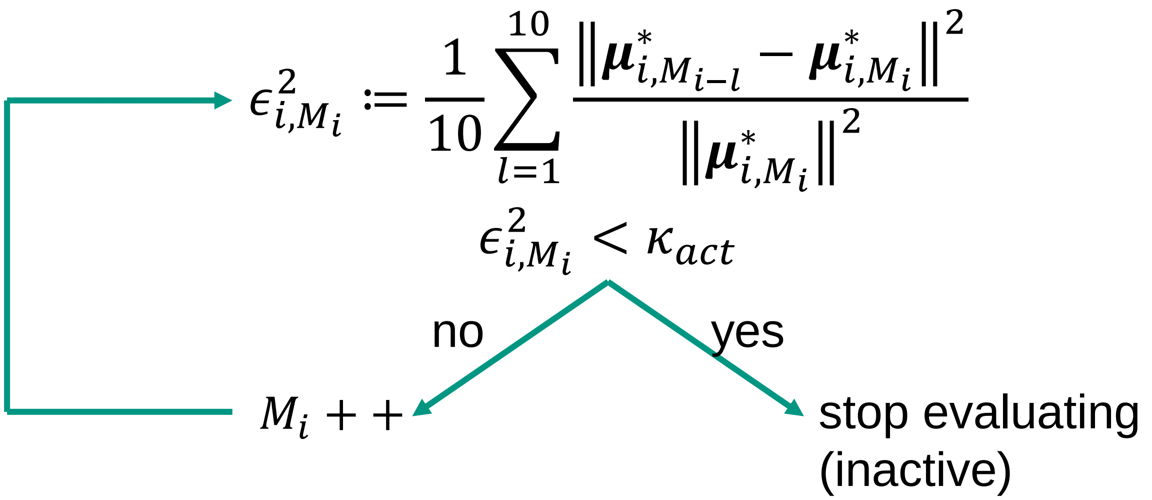
Dynamic Stop Criterion



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$$\epsilon^2_{i,M_i} < \kappa_{act}$$

no

M_i++

yes

stop evaluating input i (inactive)

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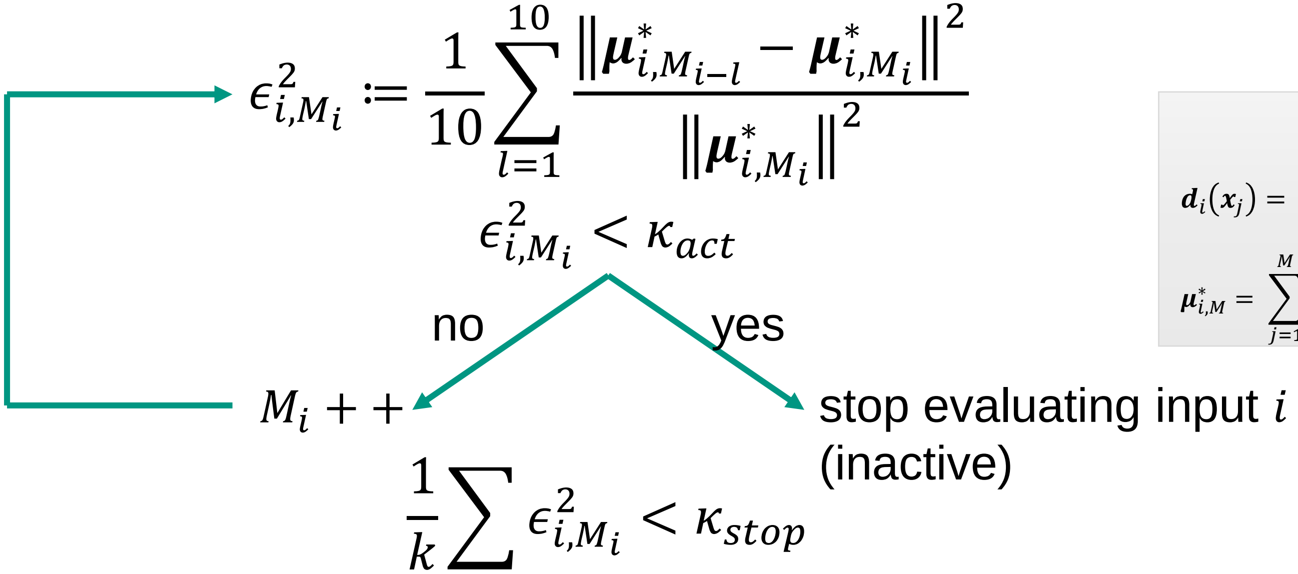
Dynamic Stop Criterion



$$\epsilon^2_{i,M_i} := \frac{1}{10}\sum_{l=1}^{10} \frac{\left\|\boldsymbol{\mu}^*_{i,M_{i-l}} - \boldsymbol{\mu}^*_{i,M_i}\right\|^2}{\left\|\boldsymbol{\mu}^*_{i,M_i}\right\|^2}$$

$$\epsilon^2_{i,M_i} < \kappa_{act}$$

no

yes

$M_i + +$

stop evaluating input i (inactive)

$$\frac{1}{k}\sum \epsilon^2_{i,M_i} < \kappa_{stop}$$

reminder:

$$\boldsymbol{d}_i(\boldsymbol{x}_j) = \frac{\boldsymbol{f}(\boldsymbol{x}_j + \Delta_i \boldsymbol{e}_i) - \boldsymbol{f}(\boldsymbol{x}_j)}{\Delta_i}$$

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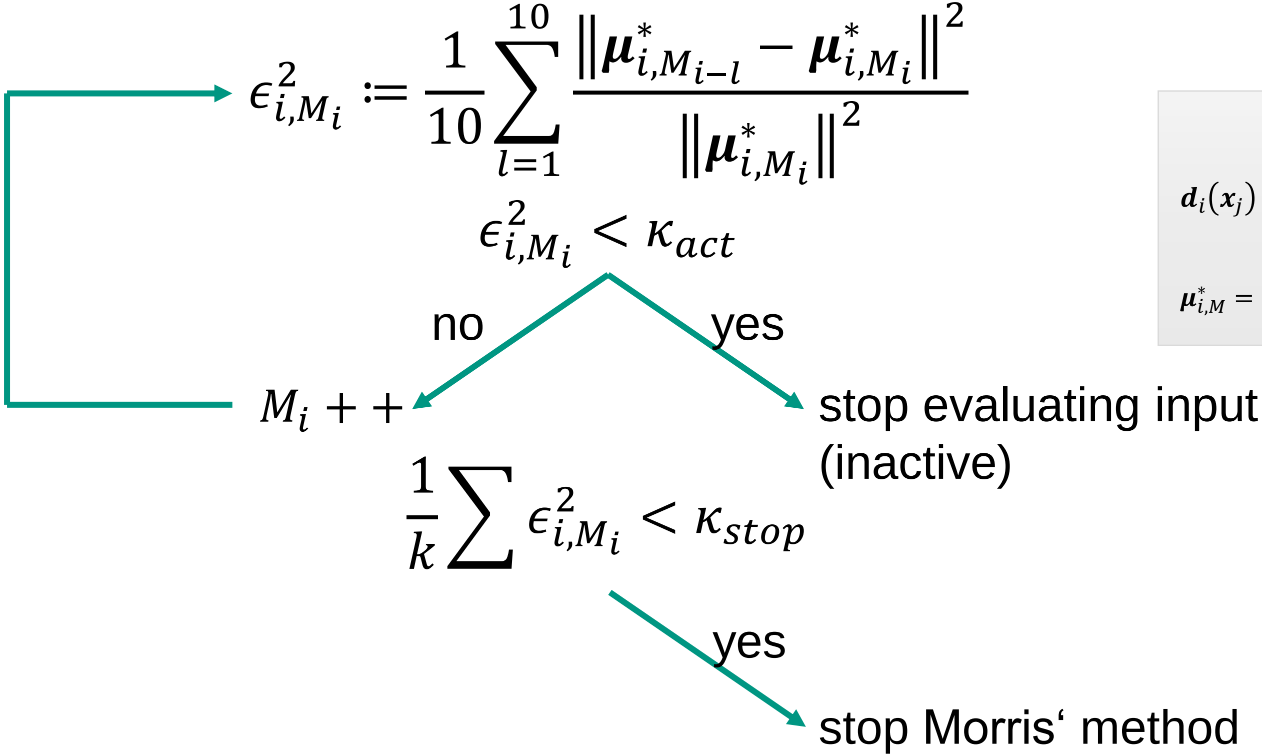

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stop Morris‘ method

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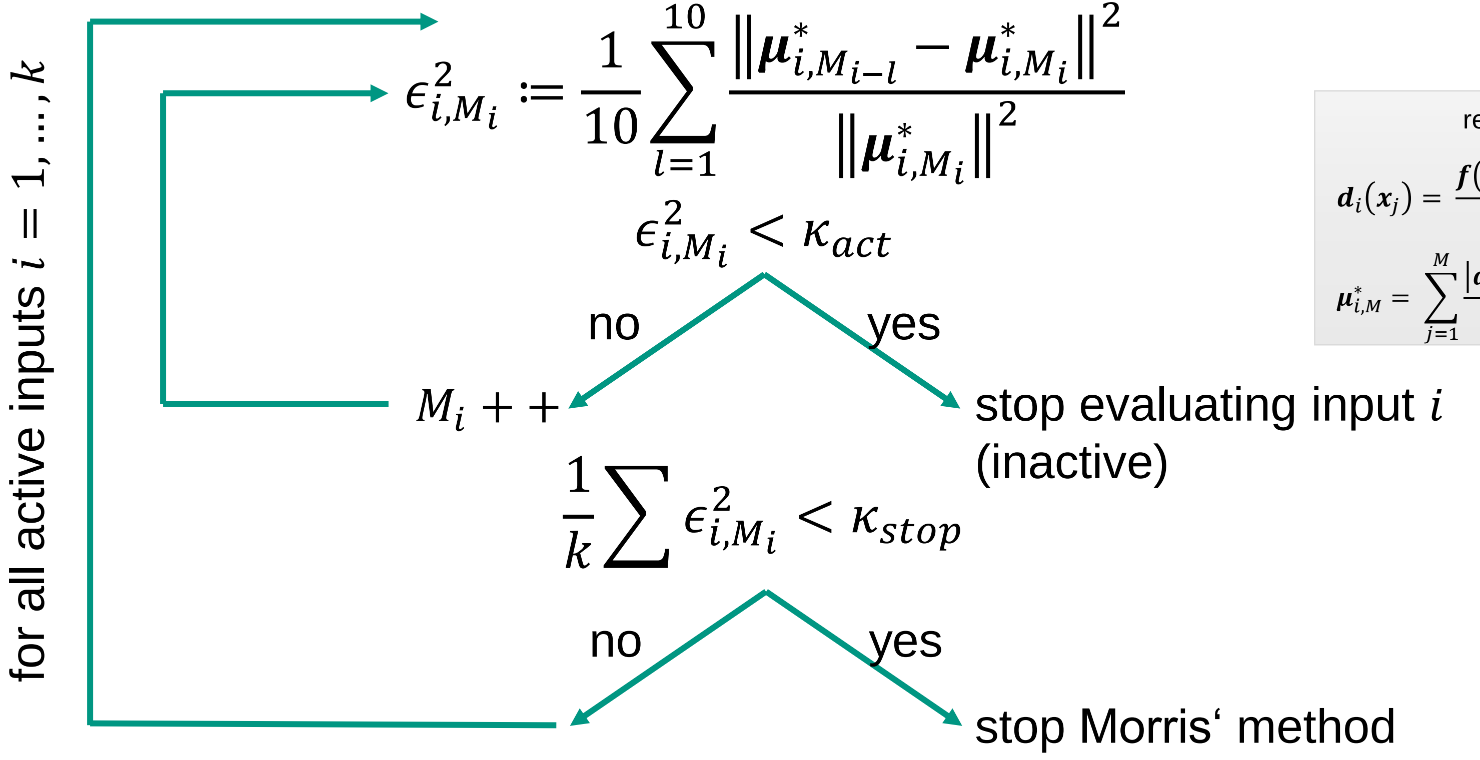

for all active inputs $i = 1, \dots, k$

$$\epsilon^2_{i,M_i} := \frac{1}{10}\sum_{l=1}^{10} \frac{\left\|\boldsymbol{\mu}^*_{i,M_{i-l}} - \boldsymbol{\mu}^*_{i,M_i}\right\|^2}{\left\|\boldsymbol{\mu}^*_{i,M_i}\right\|^2}$$

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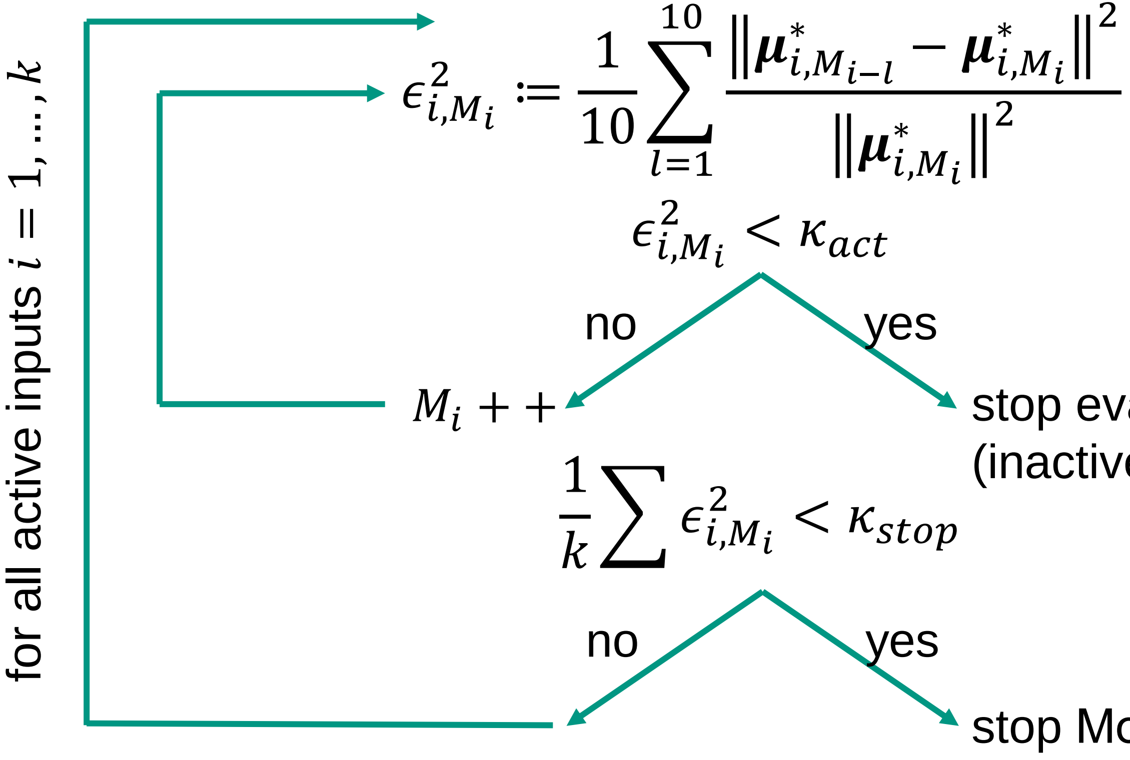

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converges because of law of large numbers ✓

Why is the dynamic stop criterion necessary?

- Morris‘ method

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$$\frac{1}{k} \sum \epsilon^2_{i,M_i} < \kappa_{stop}$$

Why is the dynamic stop criterion necessary?

- Morris‘ method
  - → qualitative method

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  - → qualitative method
  - → no exact calculation of input‘s influence

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- BUT: this is necessary for some inputs ☹

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$$d_i(x_j) = \frac{f(x_j + \Delta_i e_i) - f(x_j)}{\Delta_i}$$

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example automated driving:

```
if $\|\mu^*_{v,M_v}\| > \epsilon$ and
        $v < v_{\text{limit}}$:
    accelerate
```

Why is the dynamic stop criterion necessary?

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- However, this is important to decide if the automated vehicle broke a traffic rule

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Why is the dynamic stop criterion necessary?

- Morris‘ method
  - → qualitative method
  - → no exact calculation of input‘s influence
- BUT: this is necessary for some inputs ☹
- However, this is important to decide if the automated vehicle broke a traffic rule
  - → the dynamic stop criterion increases runtime **only if necessary**

reminder:

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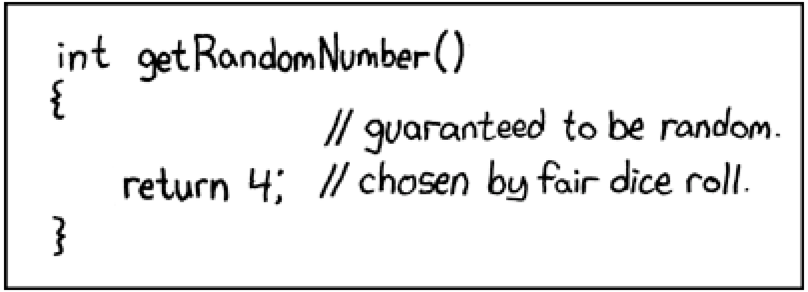
Content

1. Morris‘ Method of Elementary Effects
2. Sampling Strategy: Dynamic Stop Criterion
3. Results



```
int getRandomNumber()
{
                    // guaranteed to be random.
    return 4;  // chosen by fair dice roll.
}
```

adapted from https://xkcd.com/221/

Results: g-Function (as in Campolongo *et al*., 1997)

$$g\colon [0,1]^6 \to \mathbb{R},$$

$$\boldsymbol{x} \mapsto g(\boldsymbol{x}) := \prod_{i=1}^{6} g_i(x_i)$$

$$= \prod_{i=1}^{6} \frac{|4x_i - 2| + a_i}{1 + a_i},$$

$$\boldsymbol{x} \sim \mathcal{U}([0,1]^6)$$

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- ● $a_1 = 0,$
- ● $a_2 = 0.2,$
- ● $a_3 = 0.9,$
- ● $a_4 = 9,$
- ● $a_5 = 50,$
- ● $a_6 = 99$

relevant ↓ irrelevant

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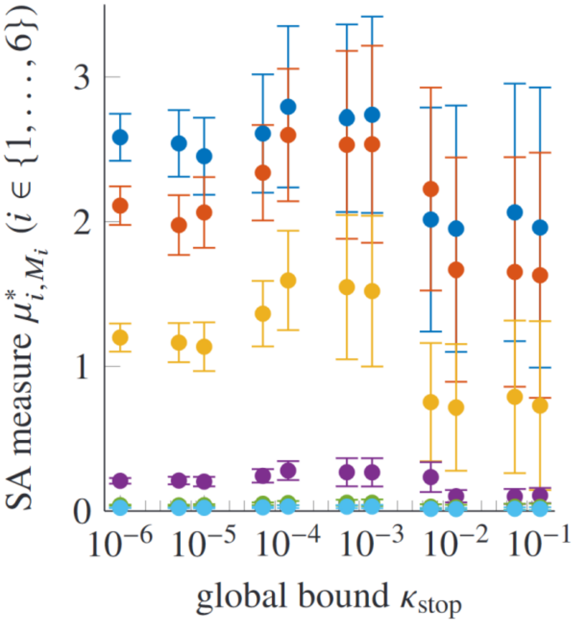

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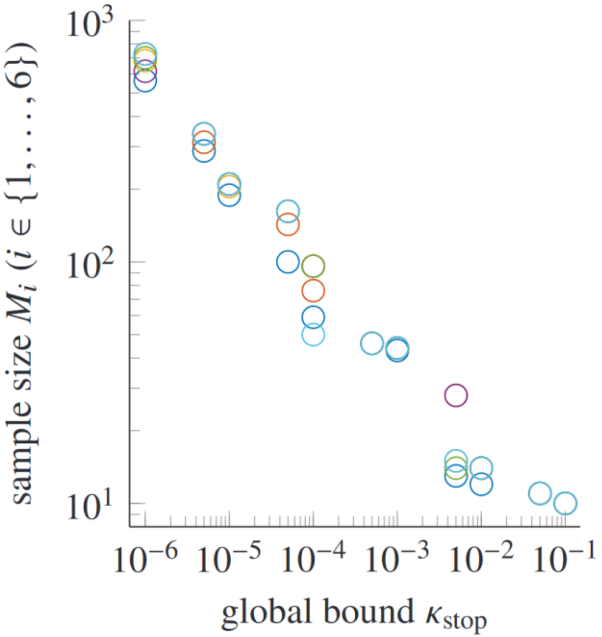
A closer look at the number of samples



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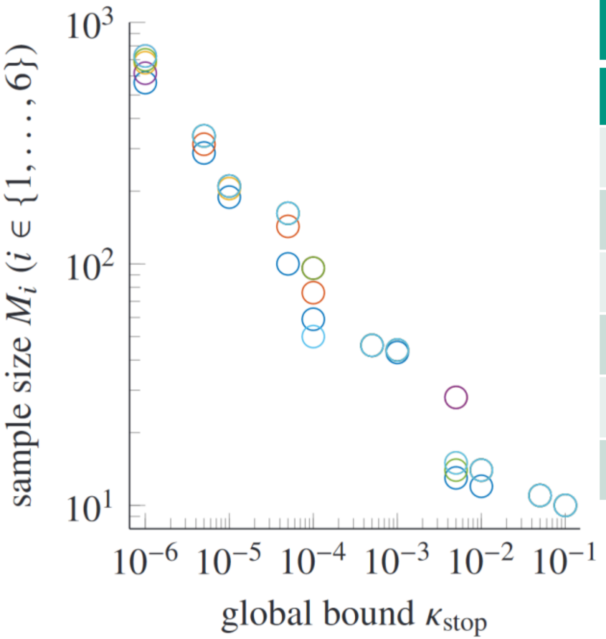
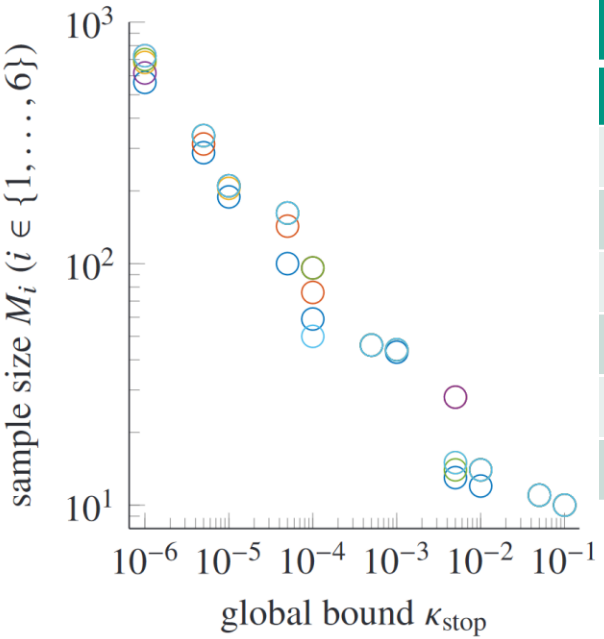


κ_{stop}	number of samples M_i for inputs $i = \{1, \ldots, 6\}$					
	● $a_1 = 0$	● $a_2 = 0.2$	● $a_3 = 0.9$	● $a_4 = 9$	● $a_5 = 50$	● $a_6 = 99$
10^{-1}	10	10	10	10	10	10
10^{-2}	**12**	14	14	14	14	14
10^{-3}	**43**	44	44	44	44	44
10^{-4}	59	76	96	96	96	**50**
10^{-5}	**189**	210	205	210	210	210
10^{-6}	**563**	726	681	616	700	726

relevant → irrelevant

A closer look at the number of samples



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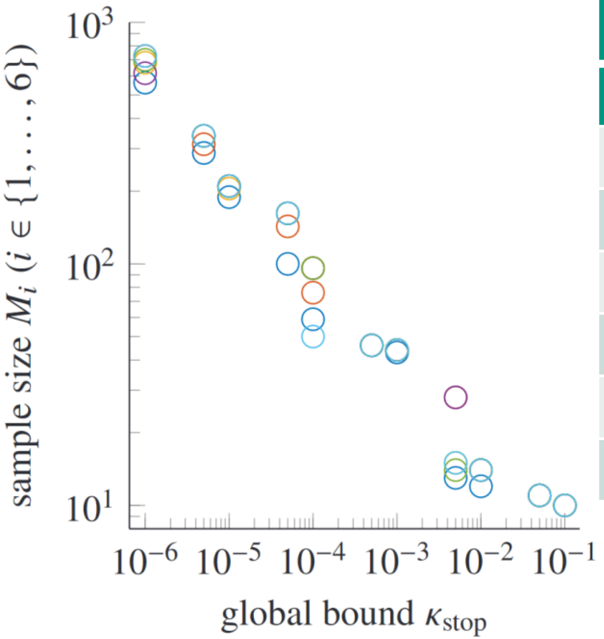

κ_{stop}	number of samples M_i for inputs $i = \{1, \dots, 6\}$						$\sum M_i$	$\frac{\sum M_i}{6 \max M_i}$
	● $a_1 = 0$	● $a_2 = 0.2$	● $a_3 = 0.9$	● $a_4 = 9$	● $a_5 = 50$	● $a_6 = 99$		
10^{-1}	10	10	10	10	10	10	60	1
10^{-2}	**12**	14	14	14	14	14	82	0.976
10^{-3}	**43**	44	44	44	44	44	263	0.996
10^{-4}	59	76	96	96	96	**50**	473	0.821
10^{-5}	**189**	210	205	210	210	210	1234	0.979
10^{-6}	**563**	726	681	616	700	726	4012	0.921

relevant → irrelevant

What did we talk about today?

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What did we talk about today?

- Morris‘ Method

What did we talk about today?

- Morris‘ Method
- sampling-based

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What did we talk about today?

- Morris‘ Method
- sampling-based
- dynamic stop criterion

What did we talk about today?

- Morris‘ Method
- sampling-based
- dynamic stop criterion
- reduce runtime

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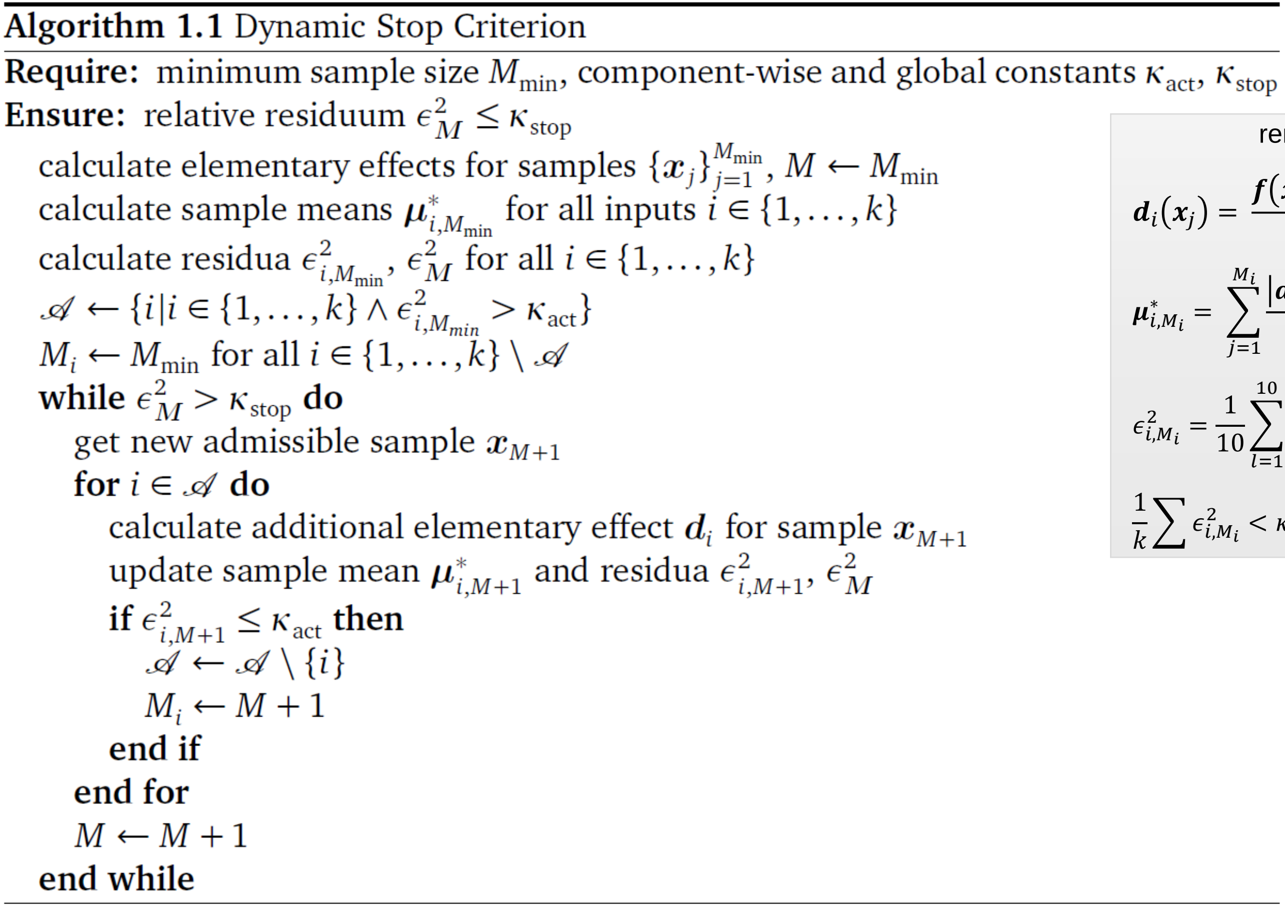
Appendix



Algorithm 1.1 Dynamic Stop Criterion

```
Require: minimum sample size $M_{\min}$, component-wise and global constants $\kappa_{\text{act}}$, $\kappa_{\text{stop}}$
Ensure: relative residuum $\epsilon_M^2 \leq \kappa_{\text{stop}}$
  calculate elementary effects for samples $\{\boldsymbol{x}_j\}_{j=1}^{M_{\min}}$, $M \leftarrow M_{\min}$
  calculate sample means $\boldsymbol{\mu}_{i,M_{\min}}^*$ for all inputs $i \in \{1,\dots,k\}$
  calculate residua $\epsilon_{i,M_{\min}}^2$, $\epsilon_M^2$ for all $i \in \{1,\dots,k\}$
  $\mathscr{A} \leftarrow \{i | i \in \{1,\dots,k\} \wedge \epsilon_{i,M_{min}}^2 > \kappa_{\text{act}}\}$
  $M_i \leftarrow M_{\min}$ for all $i \in \{1,\dots,k\} \setminus \mathscr{A}$
  while $\epsilon_M^2 > \kappa_{\text{stop}}$ do
    get new admissible sample $\boldsymbol{x}_{M+1}$
    for $i \in \mathscr{A}$ do
      calculate additional elementary effect $\boldsymbol{d}_i$ for sample $\boldsymbol{x}_{M+1}$
      update sample mean $\boldsymbol{\mu}_{i,M+1}^*$ and residua $\epsilon_{i,M+1}^2$, $\epsilon_M^2$
      if $\epsilon_{i,M+1}^2 \leq \kappa_{\text{act}}$ then
        $\mathscr{A} \leftarrow \mathscr{A} \setminus \{i\}$
        $M_i \leftarrow M+1$
      end if
    end for
    $M \leftarrow M+1$
  end while
```

reminder:

$$\boldsymbol{d}_i(\boldsymbol{x}_j) = \frac{\boldsymbol{f}(\boldsymbol{x}_j + \Delta_i \boldsymbol{e}_i) - \boldsymbol{f}(\boldsymbol{x}_j)}{\Delta_i}$$

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$$\frac{1}{k} \sum \epsilon_{i,M_i}^2 < \kappa_{stop}$$

On the minimum sample number $M_{\min}$ and the constants $\kappa_{\text{act}}, \kappa_{\text{stop}}$

- minimum number of samples $M_{\min}$
  - is influence detected?
  - if too small, influential inputs are easily overseen
  - for Central Limit Theorem: $M_{\min} \geq 30$
- global stopping constant κ_{stop}
  - stabilizes characteristic quantities $\boldsymbol{\mu}^*, \boldsymbol{\mu}, \boldsymbol{\sigma}$
  - the smaller κ_{stop}, the smaller is the change of the quantities over the last 10 iterations
- component-wise constant κ_{act}
  - the smaller it is, the more „unnecessary“ samples are evaluated

reminder:

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Relation between minimum number of samples $M_{\min}$ and $\boldsymbol{\mu}^*$