Global reliability sensitivity analysis of models with dependent inputs using failure samples

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Contribution: RSA with dependent inputs as RA byproduct

Reliability sensitivity analysis (RSA) with dependent inputs is largely unexplored. We demonstrate how to efficiently compute the indices of [3] for reliability targets as byproduct of a single reliability analysis (RA).

Reliability analysis

The failure probability of a model with input a random vector(RV) $\boldsymbol{X} \in \mathcal{X} \subseteq \mathbb{R}^d$, distribution $\boldsymbol{X} \sim f_{\boldsymbol{X}}$ and limit-state function (LSF) $g : \mathcal{X} \to \mathbb{R}$ is

$$\mathbb{P}(\mathrm{F}) = \int_{\mathcal{X}} \mathrm{I}[g(\boldsymbol{x}) \leq 0] f_{\boldsymbol{X}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \mathbb{E}[\mathrm{I}[g(\boldsymbol{X}) \leq 0]]. \quad (1)$$

By convention, we choose the LSF to describe failure as $g(\boldsymbol{x}) \leq 0$, hence the integral over $f_{\boldsymbol{X}}$ censored on the failure domain $\mathrm{F} = \{\boldsymbol{x} : g(\boldsymbol{x}) \leq 0\}$ yields the failure probability. Given an isoprobabilistic transformation $T : \mathcal{X} \to \mathbb{R}^d$ with $\boldsymbol{U} = T(\boldsymbol{X})$ and $\boldsymbol{U} \sim \mathcal{N}(\boldsymbol{0}, \mathbf{I})$ (standard-normal RV), the reliability problems reads (φ is the standard-normal PDF)

$$\mathbb{P}(\mathrm{F}) = \int_{\mathbb{R}^d} \mathrm{I}[g(T^{-1}(\boldsymbol{u})) \leq 0] \varphi(\boldsymbol{u}) \mathrm{d}\boldsymbol{u}. \quad (2)$$

Variance-based RSA

Variance-based sensitivity analysis (SA) of model output (Model $\mathcal{M} : \mathcal{X} \to \mathcal{Y}$ with $Y = \mathcal{M}(\boldsymbol{X})$) attributes fractions of the output variance $\mathbb{V}[Y]$ to the input variables $\{X_i\}_{i=1}^d$. After normalization with $\mathbb{V}[Y]$ we obtain first-order Sobol' indices as

$$S_i = \frac{\mathbb{V}[\mathbb{E}[Y|X_i]]}{\mathbb{V}[Y]}, \quad ST_i = 1 - \frac{\mathbb{V}[\mathbb{E}[Y|\boldsymbol{X}_{\sim i}]]}{\mathbb{V}[Y]}. \quad (4)$$

$\mathcal{M} = \mathrm{I}[g(\boldsymbol{X}) \leq 0]$ yields reliability-oriented indices:

$$S_i = \frac{\mathbb{V}[\mathbb{E}[\mathrm{I}[g(\boldsymbol{X}) \leq 0]|X_i]]}{\mathbb{V}[\mathrm{I}[g(\boldsymbol{X}) \leq 0]]} \overset{(1)}{=} \frac{\mathbb{V}[\mathbb{P}(\mathrm{F}|X_i)]}{\mathbb{V}[\mathrm{I}[g(\boldsymbol{X}) \leq 0]]} \quad (5)$$

for the Sobol' and likewise for the total Sobol' index. We can compute $\mathbb{P}(\mathrm{F}|\boldsymbol{X}_i)$ using Bayes' rule:

$$\mathbb{P}(\mathrm{F}|X_i) = \frac{f_{X_i}(x_i|\mathrm{F})\mathbb{P}(\mathrm{F})}{f_{X_i}(x_i)}. \quad (6)$$

With sample-based RA methods (Monte Carlo, importance sampling, subset simulation), we get $\mathbb{P}(\mathrm{F})$ along with failure samples. These can be used to estimate $f_{X_i}(x_i|\mathrm{F})$ with kernel densities (KDE) [2].

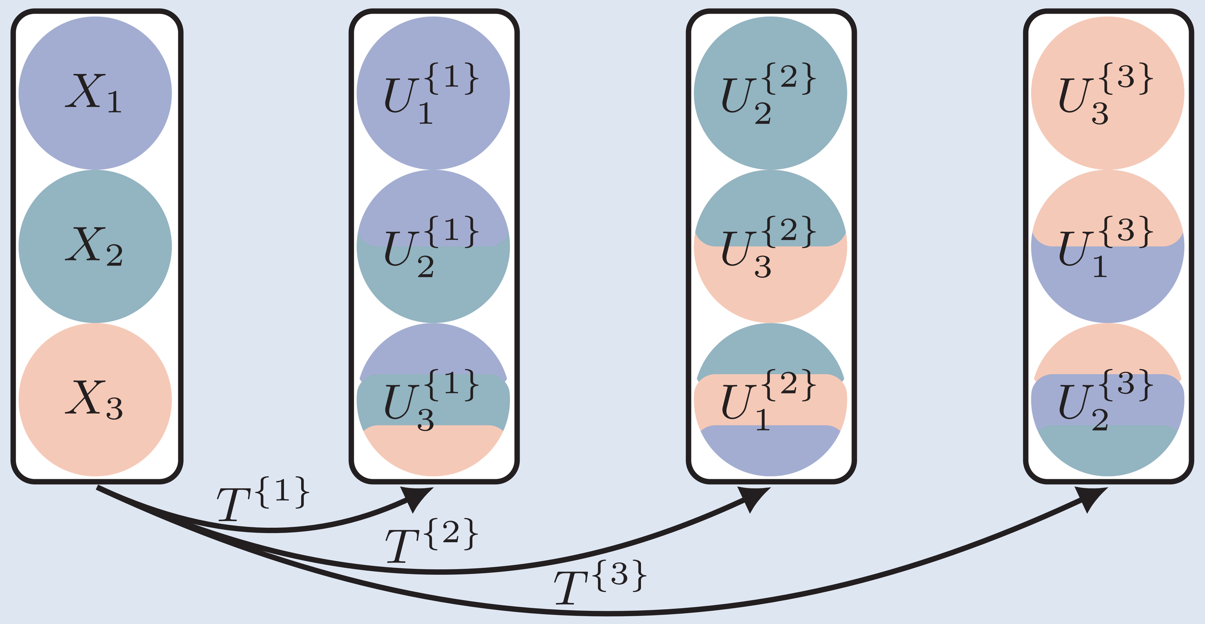
Dependent inputs

For dependent inputs $\boldsymbol{X}$, variance contributions stemming from $\mathcal{M}$ and $f_{\boldsymbol{X}}$ are difficult to discern. A possible path forward is to transform the problem to an independent probability space.

- Variance contributions can be discerned by considering different transformations T from $\mathcal{X}$ to standard-normal space.
- T is non-unique → sensitivity metrics depend on the choice of T.
- Suppose, T has a hierarchical structure (e.g., the Nataf or the Rosenblatt transform):

$$T^{\{i\}} : \begin{bmatrix} U_i^{\{i\}} \\ U_{i+1}^{\{i\}} \\ \vdots \\ U_{i-1}^{\{i\}} \end{bmatrix} = \begin{bmatrix} h_1^{\{i\}}(X_i) \\ h_2^{\{i\}}(X_i, X_{i+1}) \\ \vdots \\ h_d^{\{i\}}(X_i, X_{i+1}, \ldots, X_{i-1}) \end{bmatrix}$$

- [3] consider all d cyclic left shifts of the ordered set $\{X_1, \ldots, X_d\}$ (they use T^{-1} instead of T):



- Under $T^{\{i\}}$, X_{i-1} ($X_0 \triangleq X_d$) affects only $U_{i-1}^{\{i\}}$ and X_i affects all of $\boldsymbol{U}^{\{i\}}$, hence one defines

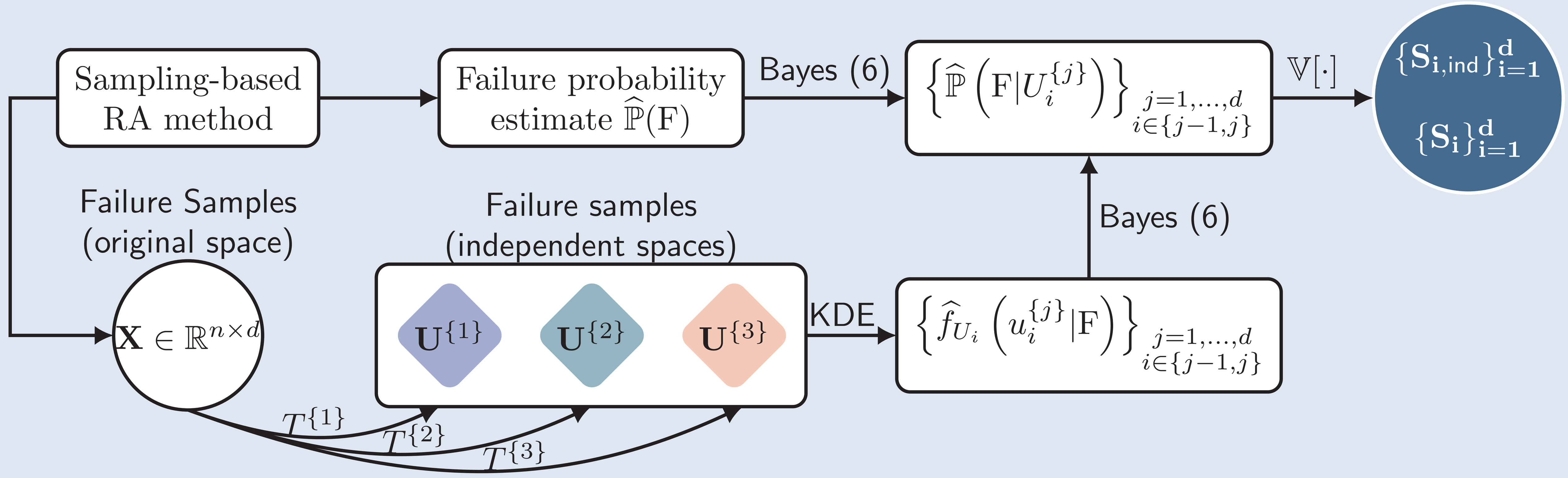
$$\Rightarrow S_{i,\mathrm{ind}} = \frac{\mathbb{V}[\mathbb{E}[Y|U_i^{\{i+1\}}]]}{\mathbb{V}[Y]}, \quad S_i = \frac{\mathbb{V}[\mathbb{E}[Y|U_i^{\{i\}}]]}{\mathbb{V}[Y]}.$$

- [3] use pick-freeze estimators in d repeated SA runs to obtain all independent and full Sobol'/total Sobol' indices.
- With n independent samples per SA run this requires a total of $nd(d+1)$ model calls.

Method

We generate a single set of failure samples $\mathbf{X}$ using a sample-based RA method. We then use each of the transformed independent sample sets $\{\mathbf{U}^{\{i\}}\}_{i=1}^d$ to estimate (similar for the total Sobol' indices):

$$S_{i,\mathrm{ind}} = \frac{\mathbb{V}[\mathbb{P}[F|U_i^{\{i+1\}}]]}{\mathbb{V}[\mathrm{I}[g(\boldsymbol{X}) \leq 0]]}, \qquad S_i = \frac{\mathbb{V}[\mathbb{P}[F|U_i^{\{i\}}]]}{\mathbb{V}[\mathrm{I}[g(\boldsymbol{X}) \leq 0]]}. \quad (3)$$



Example application

We analyze the following LSF with $\mathbb{P}(\mathrm{F}) = 6.2 \cdot 10^{-3}$:

$$g(\boldsymbol{X}) = X_1^3 + 10X_2^2 + 0.1\sin(\pi X_2) + 10X_3^2 + 40\sin(\pi X_3) + 38, \text{ where } \boldsymbol{X} \sim \mathcal{N}\left(\begin{bmatrix}0\\0\\0\end{bmatrix}, \begin{bmatrix}1 & 0.5 & 0.3\\0.5 & 1 & 0.8\\0.3 & 0.8 & 1\end{bmatrix}\right).$$

We use failure samples from subset simulation (SUS) runs with $n = 10^4$ samples per level (3 levels) and compare with [1] & [3] each using $n = 10^6$ independent samples. Statistics are based on 100 repeated runs.

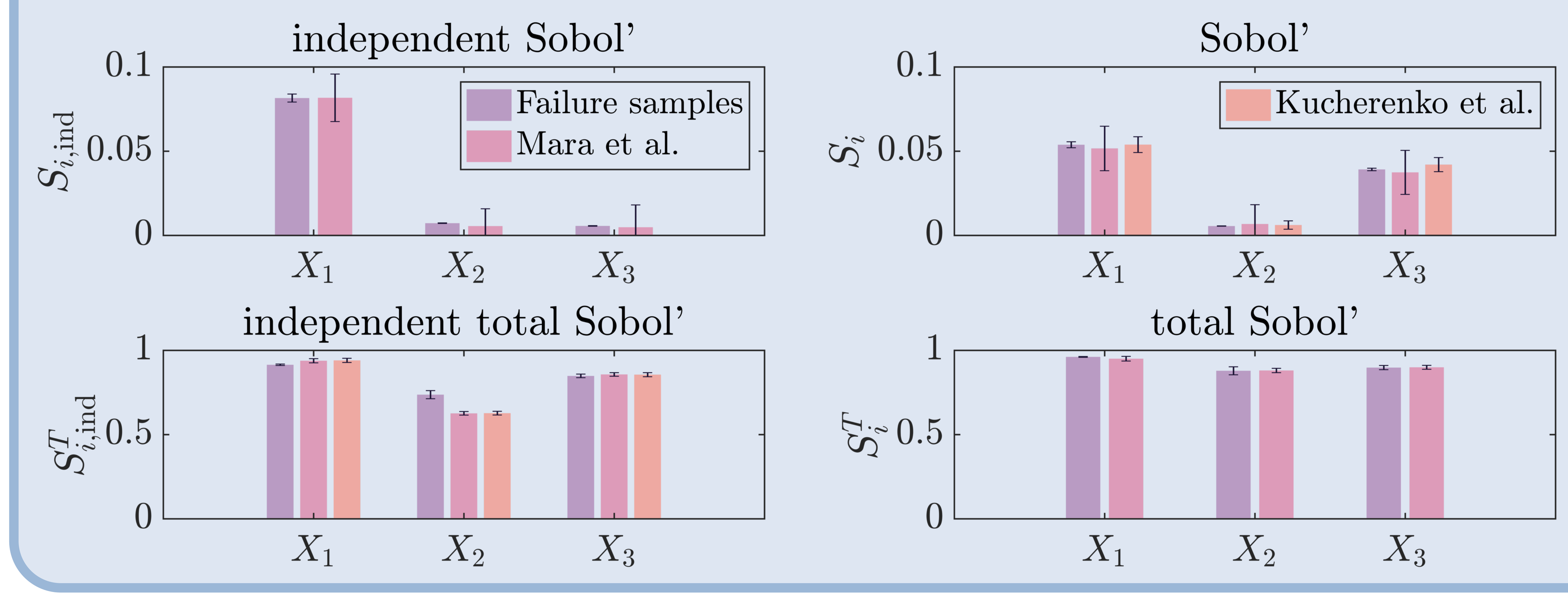


Figure 1 (left): mean estimates (bar plot) $\pm\sigma$ (black brackets).

Table 1: Computational cost.

Method	LSF calls
Mara et al.	$12 \cdot 10^6$
Kucherenko	$3 \cdot 10^6$
Failure samples	$0.03 \cdot 10^6$

Outlook

Our approach can be easily adpated to compute first-order independent and full indices of model output by using order statistics of the independent back-transformed output samplef or each transformation $T^{\{i\}}$ to computing $\mathbb{E}[Y|U_i^{\{1\}}]$.

References

[1] S. Kucherenko, S. Tarantola, and P. Annoni. Estimation of global sensitivity indices for models with dependent variables. *Computer Physics Communications*, 183(4):937–946, 2012.

[2] Luyi Li, Iason Papaioannou, and Daniel Straub. Global reliability sensitivity estimation based on failure samples. *Structural Safety*, 81:101871, 2019.

[3] Thierry A. Mara, Stefano Tarantola, and Paola Annoni. Non-parametric methods for global sensitivity analysis of model output with dependent inputs. *Environmental Modelling & Software*, 72:173–183, 2015.